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Indeterminacy of Relative Prices in
Overlapping Generations Models

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Abstract

In this paper we consider stationary pure exchange overlapping generations models with n goods in each period. We argue that for a model with non-zero stock of nominal debt there is potentially an n - 1 dimensional indeterminacy. Thus relative prices within a period can be indeterminate. Although our results agree with those previously known for the case where there is one good in every period, they indicate that indeterminacy does not depend on the existence of fiat money or other assets. Furthermore, even in pure exchange models with no aggregate debt or assets, our results indicate that equilibria may be indeterminate or not whether or not they are pareto efficient. We construct robust examples of models with indeterminacy of relative prices in which the only departure from the simple model with one consumer in every generation and one good in every period is that each consumer lives for three, rather than two, periods. This three period lived consumer model can also be viewed as a model of one with two two period lived consumers in each generation and two goods in each period.

Indeterminacy of Relative Prices in Overlapping Generations Models

by

Timothy J. Kehoe and David K. Levine*

I. Introduction

The overlapping generations model developed by Samuelson (1958) provides an attractive alternative to models with infinitely lived agents as an intertemporal general equilibrium model. In contrast to models with a finite number of infinitely lived agents, it may possess equilibria that are not pareto efficient. It may also possess equilibria in which a stock of nominal debt, often identified as fiat money when it is positive, is passed from generation to generation. These features, among others, have made this type of model popular in discussions of the theoretical aspects of such issues as social security schemes and national debt (Diamond (1965)), monetary policy (Lucas (1972)), and international exchange rates (Kareken and Wallace (1981)). This type of model would also seem to be ideal as a tool for policy analysis. Unfortunately, and also in contrast to models with a finite number of infinitely lived agents, overlapping generations models may not have determinate equilibria (Kehoe and Levine (1982a)).

That an overlapping generations model might have a continuum of equilibria is well known. When counting the equations and unknowns in his equilibrium conditions, Samuelson himself has noted that "we never seem to get enough equations: Lengthening our time period turns out always to add as

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many new unknowns as it supplies equations." Gale (1973) has extensively studied the overlapping generations model with a single two period lived consumer in each generation and one good in each period. In such a model he finds that indeterminacy is always associated with equilibria that are inefficient and have positive amounts of nominal debt or with equilibria that are efficient and have negative amounts of nominal debt. In either case there is a one dimensional set of equilibria; in other words, the equilibria can be indexed by a single number, for example, the price of fiat money. Balasko and Shell (1981) have extended these results to a model in which there are many goods in each period, but a single two period lived consumer in each generation, in fact, one with a Cobb-Douglas utility function. Calvo (1978) has constructed examples in which the indeterminacy is still one dimensional, indexed by the price of an asset such as land or capital.

In this paper we consider pure exchange overlapping generations models with n goods in each period. We argue that for a model with a non-zero stock of nominal debt there is potentially an n dimensional indeterminacy, while for a model with no nominal debt there is potentially an n - 1 dimensional indeterminacy. Thus relative prices within a period can be indeterminate. Although our results agree with those previously known for the case where there is one good in every period, they indicate that indeterminacy does not depend on the existence of fiat money or other assets. Furthermore, even in pure exchange models with no aggregate debt or assets, our results indicate that equilibria may be indeterminate or not whether or not they are pareto efficient.

How far do we have to go to construct examples in which there are indeterminate equilibria without fiat money or indeterminate equilibria that are pareto efficient? We shall present an example in which the only

departure from the simple model considered by Gale is that the single consumer in each generation lives three, rather than two periods. Gale himself considers such models and conjectures that the results he obtains for the two period lived model carries over to them. Unfortunately, we provide a robust example that demonstrates that this is not the case. This stands in fundamental constrast to the static pure exchange model, where, although it is always possible to construct examples with a continuum of equilibria, such examples cannot be robust. As we shall see, our three period lived consumer model can also be viewed as a model with two two period lived consumers in each generation and two goods in each period.

We begin by describing a simple stationary model and examining its steady states. We then study the behavior of equilibrium price paths around a steady state and characterize the dimensionality of paths that converge to the steady state. We also indicate how our results can be extended to models that have growing populations and that are non-stationary for a finite number of periods. Finally, we discuss the significance of indeterminacy in models of this type.

II. The Model and Its Steady States

We begin by considering the model in which each generation lives two periods. As we shall explain, the model in which each generation lives more than two periods can be viewed as a special case of this model. Each generation $t > 1$ is identical and lives in periods t and $t + 1$. There are n goods in each period. The vector $p_t = (p_t^1, \ldots, p_t^n)$ denotes prices in t. The consumption and savings decisions of the (possibly many different types of) consumers in generation t are aggregated into excess demand functions

$y(p_t, p_{t+1})$ when young and $z(p_t, p_{t+1})$ when old; y and z are, of course, n dimensional vectors. Excess demands are assumed to be homogeneous of degree zero,

$$y(\theta p_t, \theta p_{t+1}) \equiv y(p_t, p_{t+1})$$
$$z(\theta p_t, \theta p_{t+1}) \equiv z(p_t, p_{t+1}) \tag{1}$$

for any $\theta > 0$, and to satisfy Walras's law, the aggregate demand budget constraint,

$$p_t'y(p_t, p_{t+1}) + p_{t+1}'z(p_t, p_{t+1}) \equiv 0. \tag{2}$$

Both of these assumptions can be justified if each consumer faces a single budget constraint, in other words, if we allow consumers to make intertemporal trades.

We assume that excess demands are continuously differentiable for all strictly positive price pairs (p_t, p_{t+1}), which, as Debreu (1972) and Mas-Colell (1974) have shown, entails little loss of generality. We further assume that y and z are bounded from below and such that as some, but not all, prices approach zero, $e'(y(p_t, p_{t+1}) + z(p_t, p_{t+1})) \to \infty$ where e denotes the n vector whose every element is one. These assumptions are naturally satisfied if y and z are derived from utility maximization: If consumption of every good by every consumer must be non-negative, then an obvious lower bound for (y, z) is $(-w_1, -w_2)$ where w_1 and w_2 are the aggregate endowment vectors when young and when old respectively. If preferences are monotonically increasing in consumption, then when a single price goes to

zero the excess demand for that good becomes infinite. Furthermore, if more than one price goes to zero, then excess demand for some, but perhaps not all, of the corresponding goods becomes infinite (see Arrow and Hahn (1971), pp. 29-31.)

Debreu (1974) has demonstrated that, for any y and z that satisfy the assumptions of homogeneity and Walras's law, there exists a generation of $2n$ utility maximizing consumers whose aggregate excess demand functions y^* and z^* agree with y and z on any set of positive relative prices uniformly bounded away from zero. There is a minor technical complication in that y^* and z^* may not agree with y and z as some relative prices approach zero. Utilizing a result due to Mas-Colell (1977), however, Kehoe and Levine (1982c) argue that we can ignore this qualification when studying the behavior of the excess demand functions near steady states. Consequently, for our purposes, we are justified in viewing our assumptions as both necessary and sufficient for demand functions derived from utility maximization by heterogeneous consumers. As we shall see, however, the possibility of indeterminacy of relative prices in overlapping generations models does not depend on implausible aggregate excess demand functions.

The assumptions of homogeneity and Walras's law are implicitly equivalent to the assumption that consumers are allowed to trade goods with each other even if the goods are consumed in different time periods. One institutional story to go with this assumption is that we allow creation of private debt, or inside money. The presence of a public debt, or outside money, is a different matter, however, which depends only on initial conditions. An equilibrium price path for this economy is one for which excess demand vanishes in each period:

$$z_0(p_1) + y(p_1, p_2) = 0 \tag{3}$$

for t = 1 and

$$z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0 \tag{4}$$

for $t > 1$. Here z_0 denotes the excess demand of old people in the first period. Let $\mu = p_1' z_0(p_1)$ be the nominal savings of the initial old generation. Repeated application of the equilibrium conditions and Walras's law implies that $-p_t' y(p_t, p_{t+1}) = p_{t+1}' z(p_t, p_{t+1}) = \mu$ at all times. Consequently, we can view μ as a constant stock of outside money, fiat money, at least if $\mu > 0$. It is the fixed nominal net savings done by the young generation each period.

A steady state of this economy is a relative price vector p and an inflation factor β such that $p_t = \beta^t p$ satisfies

$$z(\beta^{t-1}p, \beta^t p) + y(\beta^t p, \beta^{t+1}p) = z(p, \beta p) + y(p, \beta p) = 0. \tag{5}$$

In other words, if relative prices in each period are given by p and the price level grows by a factor β, markets would necessarily clear in all but possibly the first period. Since claims to good i now cost p^i and claims next period cost βp^i, $1/\beta - 1$ is the steady state rate of interest.

There are two types of steady states: real steady states in which $\mu = -p'y(p, \beta p) = 0$ and monetary, or nominal, steady states in which $\mu \neq 0$. On one hand, Walras's law implies that $p'(y + \beta z) = 0$ and, consequently, that $\beta p'z = \mu$. On the other, the equilibrium condition (5) implies that

$p'(y + z) = 0$ and, consequently, that $p'z = \mu$. Therefore $(\beta - 1)\mu = 0$, and any nominal steady state must have $\beta = 1$. Gale calls steady states in which $\beta = 1$ golden rule steady states because they maximize a weighted sum of individual utility functions subject to the constraint of stationary consumption over time. He calls real steady states balanced.

It is possible to construct examples in which a golden rule steady state is also balanced, in other words, in which $\mu = 0$ and $\beta = 1$ simultaneously. Such a steady state must satisfy $z(p, p) + y(p, p) = 0$ and $-p'y(p, p) = 0$. Walras's law implies that this is a system of n independent equations; homogeneity implies that there are n - 1 independent unknowns. Consequently, we would expect this system of equations to have a solution only by coincidence. In fact, Kehoe and Levine (1982c) prove that almost all economies do not have a steady state where both $\mu = 0$ and $\beta = 1$. They give the space of economies (y, z) that satisfy the assumptions of differentiability, homogeneity, Walras's law, and the boundary condition a topological structure: Two economies are close to each other if the values of the demand functions and the values of their partial derivatives are uniformly close. The phrase "almost all" in this context means that the property holds for a subset that is open and dense: Any sufficiently small perturbation of an economy that does not have a steady state where $\mu = 0$ and $\beta = 1$ results in an economy that still does not have such a steady state; for any economy that has such a steady state, however, there exist arbitrarily small perturbations that result in economies that do not have such steady states. A property that holds for almost all economies is called a generic property.

Gale proves that the model with a single two period lived consumer in each generation has a unique nominal steady state and, generically, a unique real steady state. The unique nominal steady state is where the price of the

single good is constant over time. Walras's law implies that this situation does indeed satisfy the steady state condition (5). At this steady state the savings of the young person are not, in general, zero. Since there is only one consumer in each generation any trade that takes place must be between generations. Consequently, since there is only one good in each period, there can be trade only if there is a corresponding transfer of nominal debt from period to period. Any real steady state must, therefore, by given by a relative price ratio $\beta = p_{t+1}/p_t$ that makes the consumer prefer not to trade. Such a price ratio obviously exists; generically there is only one.

With many consumers in each generation, but only one good in each period, nominal steady states are still unique, but real steady states need not be: Consider a one period pure exchange economy with two consumers and two goods that has multiple, but determinate, equilibria. Robust examples of this sort are, of course, easy to construct (see, for example, Shapley and Shubik (1973)). Now construct an overlapping generations economy by assigning two such consumers to each generation and by letting one of the goods be available in the first period of their lives and the other in the second. Each of the different equilibria of the static economy now corresponds to a different real steady state of the overlapping generations economy in which the two consumers in each generation trade with each other, but not with other generations. With many goods and many consumers neither real steady states nor nominal steady states need be unique. Kehoe and Levine (1982c) prove, however, that generically there exists an odd number of each type. Their arguments are similar to those used to prove that the number of equilibria of a pure exchange economy is odd. (See, for example, Varian (1974).)

Using a result due to Balasko and Shell (1980), we are able to examine

the efficiency properties of steady states. They consider models with a single consumer in each generation that satisfy a uniform curvature condition on indifference surfaces that is quite natural in a stationary setting such as ours. They demonstrate that a necessary and sufficient condition for an equilibrium price path of such a model to be pareto efficient is that the infinite sum $\Sigma 1/||p_t||$ does not converge. (Here, of course, $||p_t|| = (p_t'p_t)^{1/2}$.) This result can easily be extended to models with many consumers in every generation. Consequently, a steady state of our model is pareto efficient if and only if $\beta \leq 1$, in other words, if and only if the interest rate is non-negative. Price paths that converge to steady states where $\beta \leq 1$ are pareto effficient; those that converge to steady states where $\beta > 1$ are not.

Every economy has a pareto efficient steady state since it always has a steady state where $\beta = 1$. In the model with one two period lived consumer in each generation and one good in each period, Gale finds that the unique real steady state has $\beta < 1$ if and only if the unique nominal steady state has $\mu < 0$. Similarly, $\beta > 1$ at the real steady state if and only if $\mu > 0$ at the nominal steady state. In the more general model we cannot make such strong statements. We can, however, demonstrate that every economy has an odd number of steady states where $\beta \leq 1$ and $\mu \leq 0$. We sketch an argument below; details are given by Kehoe and Levine (1982c). This argument also makes it clear why every economy has an odd number of real steady states and an odd number of nominal steady states.

Consider the n functions

$$\bar{f}_i(p, \beta) = y^i(p, \beta p) + z^i(p, \beta p) - p'(y(p, \beta p) + z(p, \beta p)). \qquad (6)$$

Our assumption on the behavior of excess demand as some prices tend toward zero guarantees that there exists $\underline{\beta} > 0$ and $\overline{\beta} > \underline{\beta}$ such that any steady state value of β satisfies $\underline{\beta} < \beta < \overline{\beta}$. Furthermore, for any $\beta > 0$ there exists at least one value of p that solves the equations

$$f_i(p, \beta) = 0, \quad i = 1, \ldots, n - 1. \tag{7}$$

Since this is a system of n - 1 equations in n - 1 unknowns it can easily be shown that solutions to this system of equations are generically smooth implicit functions of β.

There are two distinct ways for a pair (p, β) that satisfies (7) to be a steady state: if $-p'y(p, \beta) = 0$ or if $\beta = 1$. In either case Walras's law implies that $p'(y(p, \beta p) + z(p, \beta p)) = 0$. Let

$$m(p, \beta) = -p'y(p, \beta p) \tag{8}$$

for any (p, β), and consider pairs μ and β such that $\mu = m(p, \beta)$ and (p, β) satisfies (7). Unfortunately, μ is not, in general, a well defined function of β since for any β there may be more than one p such that (7) is satisfied. We are justified, however, in drawing diagrams such as that in Figure 1.

Figure 1

There are a finite number of paths of pairs μ and β that satisfy our

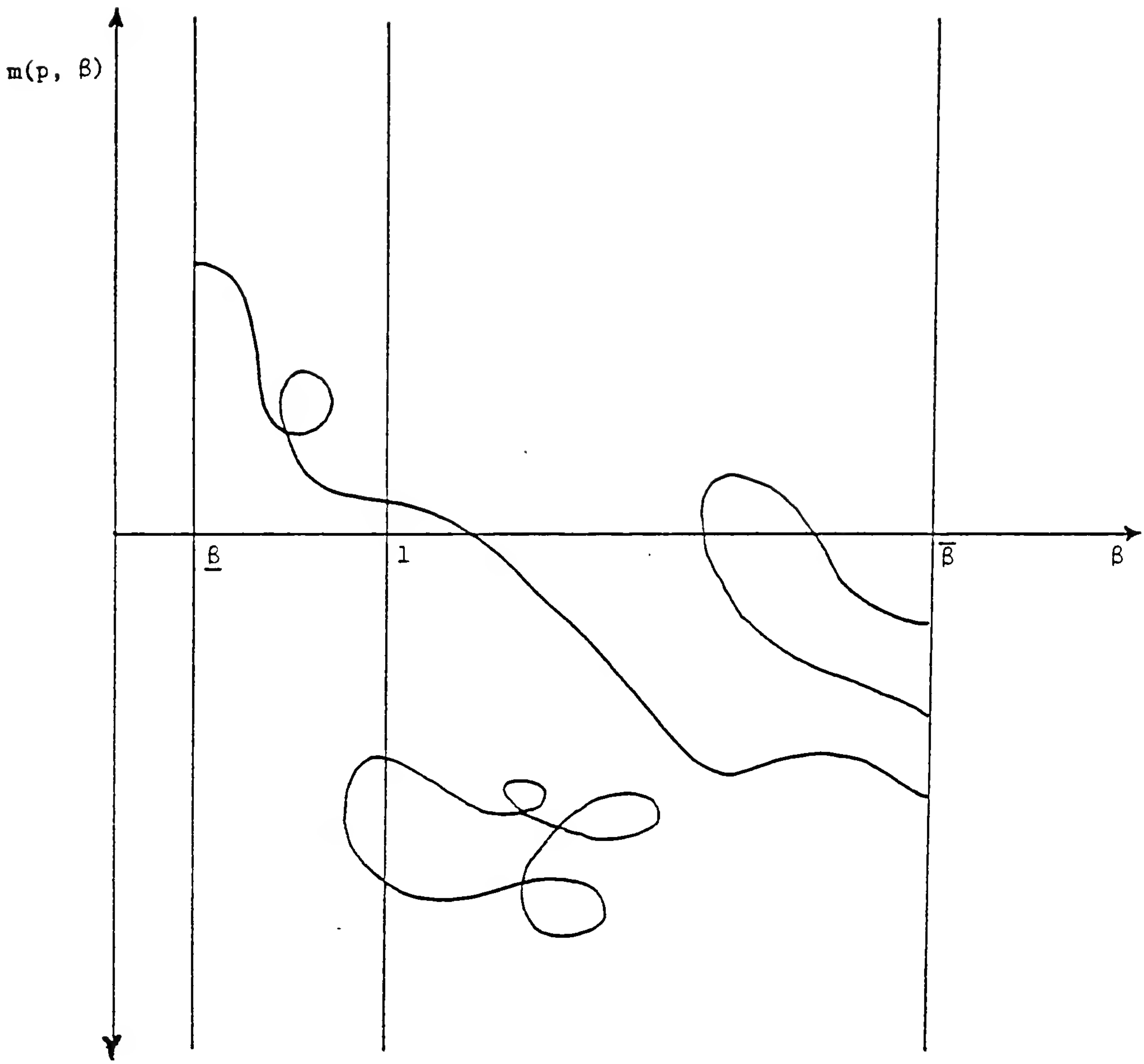

Figure 1

conditions. Some of them are loops that do not intersect the boundary $\beta = \underline{\beta}$ or $\beta = \overline{\beta}$. It is possible, however, to demonstrate that generically there are an odd number of points of the form $(p, \underline{\beta})$ where (7) is satisfied, and, similarly, an odd number of the form $(p, \overline{\beta})$. To make this plausible observe that, for any fixed β, the functions $f_i(p, \beta)$ have the formal properties of excess demand functions of a static pure exchange economy: They are homogeneous of degree zero in p and satisfy Walras's law, $p'f(p, \beta) = 0$. It is well known that generically (in this case, for almost all β) there are an odd number of equilibria of such economies. An even number, possibly zero, of pairs μ and β are associated with paths that return to the boundary $\beta = \underline{\beta}$. An odd number, at least one, therefore cannot return. Our boundary assumption implies that, for $\underline{\beta} < \beta < \overline{\beta}$, the prices that satisfy (7) are uniformly bounded away from zero. Consequently, $m(p, \beta)$ remains bounded, and paths that start at $\beta = \underline{\beta}$ and do not return must eventually reach the boundary $\beta = \overline{\beta}$. Any path or loop may intersect itself, but does not, in general, do so where $\beta = 1$ (or $\underline{\beta}$ or $\overline{\beta}$) or where $\mu = 0$.

Our boundary assumption implies that $\mu > 0$ when $\beta = \underline{\beta}$ and $\mu < 0$ when $\beta = \overline{\beta}$. Consequently, any path that starts at $\underline{\beta}$ and ends at $\overline{\beta}$ must intersect the line $m = 0$ an odd number of times. Similarly, any such path must intersect the line $\beta = 1$ an odd number of times. On the other hand, every loop or path that starts and ends at the same boundary intersects both $\mu = 0$ and $\beta = 1$ an even, possibly zero, number of times. Each of these intersections corresponds to a steady state. Generically, there is none where $\mu = 0$ and $\beta = 1$. Experimenting with different possibilities we can easily verify that any admissible graph must share with that in Figure 1 the property that there are an odd number of steady states where $\beta < 1$ and $\mu > 0$ and an odd number where $\beta > 1$ and $\mu < 0$.

III. Determinacy of Equilibrium Price Paths

We now focus our attention on the behavior of equilibrium price paths near a steady state. In addition to the requirement that markets clear in every period we require that prices converge to the steady state, in other words, that $(p_t, p_{t+1})/||(p_t, p_{t+1})|| \to (p, \beta p)/||(p, \beta p)||$ as $t \to \infty$. We do this for two reasons. First, price paths that converge to a steady state are the most plausible perfect foresight equilibria: Agents can compute future prices using only local information. If prices are not converging to a steady state, however, then agents need global information to compute future prices. Second, these price paths are the easiest to study. To determine the qualitative behavior of price paths near a steady state we can linearize the equilibrium conditions. Paths that do not converge may display very complex periodic, or even chaotic, behavior. Some paths may even lead to prices that are zero or negative, where excess demands explode.

Determinacy of equilibrium price paths that converge to a steady state may still leave room for indeterminacy. There may be paths that do not converge to a steady state, but nevertheless always remain strictly positive and are, therefore, legitimate equilibria. That a model has a determinate path that converges to a steady state is a weak test. We shall establish, however, that there are robust examples of economies that fail even this test.

Consider again the equations that an equilibrium price path must satisfy:

$$z_0(p_1) + y(p_1, p_2) = 0 \qquad (3)$$

for t = 1 and

$$z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0 \tag{4}$$

for $t > 1$. Once p_1 and p_2 are determined (4) acts as a non-linear difference equation determining the rest of the price path. We begin by asking how many pairs (p_1, p_2) give rise to a price path that converges to a steady state $(p, \beta p)$. The stable manifold theorem from the theory of dynamical systems, described, for example, by Irwin (1980), implies that generically these questions can be answered by linearizing (4) around $(p, \beta p)$. We then ask how many pairs (p_1, p_2) are consistent with equilibrium in the first period. This question can be answered by linearizing (3). Pairs (p_1, p_2) that lie in the intersection of these two sets correspond to equilibrium price paths. The dimension of this intersection can generically be deduced from a simple counting argument. If this dimension is greater than zero, there is a continuum of equilibrium price paths. If it is greater than one, relative prices are indeterminate: Not even by exogenously specifying the price level can we make price paths determinate. Details of the arguments presented below are given by Kehoe and Levine (1982a).

Making use of the fact that derivatives of excess demand are homogeneous of degree minus one, we can write the linearized system as

$$D_2y\ p_2 + (D_1y + Dz_0)p_1 = Dz_0\ p \tag{3'}$$

$$D_2y\ p_{t+1} + (D_1y + \beta D_2z)p_t + \beta D_1z\ p_{t-1} = 0 \tag{4'}$$

where D_1y is, for example, the matrix of partial derivatives of y with

respect to its first vector of arguments and where all derivatives are evaluated at $(p, \beta p)$.

Kehoe and Levine (1982c) have shown that D_2y is generically non-singular. Consequently, (4') can be solved for an explicit second order difference equation. Using a standard trick, we can write this equation as the first order system $q_t = Gq_{t-1}$ where $q_t = (p_t, p_{t+1})$ and

$$G = \begin{bmatrix} 0 & I \\ -\beta D_2y^{-1}D_1z & -D_2y^{-1}(D_1y + \beta D_2z) \end{bmatrix}. \tag{9}$$

The stability properties of this difference equation are governed by the eigenvalues of G. By differentiating the homogeneity assumption we can establish that β is an eigenvalue of G since

$$G\begin{bmatrix} \beta p \\ p \end{bmatrix} = \begin{bmatrix} \beta^2 p \\ \beta p \end{bmatrix}. \tag{10}$$

Similarly differentiating Walras's law, we can establish that unity is another eigenvalue since

$$p'[-\beta D_1z \quad D_2y]G = p'[-\beta D_1z \quad D_2y]. \tag{11}$$

In the case where $\beta = 1$ these are generically the same restriction, and we have information about only one eigenvalue.

Consider now the diference equation $q_t = (1/\beta)G_{t-1}$. Any steady state $q = (p, \beta p)$ is a stationary solution to this equation. Let n^s be the number of eigenvalues of $(1/\beta)G$ that lie inside the unit circle in the complex plane, that is, whose moduli are less than unity. These correspond to

eigenvalues of G that lie inside the circle of radius β. The set of initial conditions $q_1 = (p_1, p_2)$ such that $q_t = Gq_{t-1}$ satisfies $q_t/||q_t|| \to q/||q||$ as $t \to \infty$ forms an $n^s + 1$ dimensional subspace V_s of R^{2n}. The extra dimension shows up because of homogeneity: If q_1 satisfies $q_t/||q_t|| \to q/||q||$, then so does Θq for any scalar Θ. This subspace is spanned by the n^s eigenvalues of G associated with the eigenvalues that lie inside the circle of radius β and the eigenvector q associated with the eigenvalue β.

Besides yielding a path that converges to the steady state ray, (p_1, p_2) must also satisfy the linearized equilibrium conditions in the first period, (3'). Let us first examine the situation where $\mu \neq 0$. Assume that $p_1'z_0(p_1) = \mu$ for all p_1, so that the initial nominal savings of old people are independent of prices. In this case z_0 cannot be homogeneous of degree zero in p_1, and, since $p_1'Dz_0(p_1)p_1 = -p_1'z_0(p_1) = -\mu \neq 0$, $Dz_0\ p \neq 0$. Consequently, (3') defines an n dimensional affine subset of prices (p_1, p_2) consistent with equilibrium in the first period. The intersection of this subset with the subspace of prices that yield a path that converges to the steady state generically has dimension $(n^s + 1) + n - 2n = n^s + 1 - n < n$.

There are several cases of interest: First, if $n^s < n - 1$, then generically there are no equilibrium paths that converge to this steady state. We call such a steady state unstable. Second, if $n^s = n - 1$, then stable equilibrium price paths are locally unique and, in a small enough neighborhood of the steady state, actually unique. We call such a steady state determinate. Third, if $n^s < n - 1$, then there is a continuum of locally stable paths. In fact, the (p_1, p_2) that generate these paths form a manifold of dimension $n^s + 1 - n$. We call such a steady state indeterminate. The $n^s + 1 - n$ subspace of the corresponding linear system is, in fact, the

tangent space to this manifold at $(p, \beta p)$, in other words, its best linear approximation.

Let us now consider the situation where $\mu = 0$. Here it is natural to assume that z_0 is homogeneous of degree zero in p_1, which implies that $Dz_0\, p = 0$. There are two considerations that reduce the dimension of the subspace of initial conditions that we are concerned with. First, since the equilibrium conditions (3') and (4') are now homogeneous, we can impose a price normalization and work in a $2n - 1$ dimensional affine subset of R^{2n}, for example, by setting $p_1^1 = 1$. This allows us to ignore the eigenvalue β associated with the eigenvector q. Second, since $\mu = 0$, the initial price pair (p_1, p_2), and all subsequent pairs (p_t, p_{t+1}), must satisfy $p_1'y(p_1, p_2) = 0$. This restriction can be linearized at $(p, \beta p)$ as

$$(y' + p'D_1y)p_1 + p'D_2y\, p_2 = 0. \tag{12}$$

Since D_2y is generically non-singular, this defines a $2n - 2$ dimensional subset of the normalized price space. Differentiating Walras's law with respect to p_1, we establish that $y' + p'D_1y + \beta p'D_1z = 0$ at $(p, \beta p)$. Consequently, (12) can be rewritten as

$$p'[-\beta D_1z \quad D_2y] \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0. \tag{13}$$

This implies that the unit eigenvalue associated with the (left) eigenvector $p'[-\beta D_1z \quad D_2y]$ is irrelevant on any path where $\mu = 0$.

Let $\overline{n}^s$ denote the number of eigenvalues of $(1/\beta)G$ that lie inside the unit circle excluding the root $1/\beta$, which we have argued is irrelevant. The set of prices $q_1 = (p_1, p_2)$ that satisfy (13) and the price normalization and

give rise to a price path that converges to $q = (p, \beta p)$ forms an $\overline{n}^s$ dimensional set. The set of prices $q_1 = (p_1, p_2)$ that satisfy (13) and the price normalization and are consistent with equilibrium in the first period forms an $n - 1$ dimensional set. Equilibrium price paths are associated with points in the intersection of these two sets, which generically has dimension $\overline{n}^s + (n - 1) - (2n - 2) = \overline{n}^s + 1 - n < n - 1$.

Although the eigenvalue $1/\beta$ is irrelevant for price paths in which $\mu = 0$, it is crucial for the behavior of paths where $\mu \neq 0$: If $\mu \neq 0$ initially, then the price path cannot converge to a steady state where $\beta < 1$ since $1/\beta$ is an unstable root. In other words, with a positive interest rate money cannot have value. Although paths with $\mu \neq 0$ can converge to steady states where $\beta > 1$, asymptotically the constant nominal money stock disappears because of inflation.

A warning should be given about the generic nature of our results. Although they hold for almost all economies, it is possible to think of examples that violate them: When there is a single two period lived consumer with an intertemporally separable utility function in each generation, for example, both D_2y and D_1z have rank one. Consequently if there are two or more goods, we cannot invert D_2y. In this case, Kehoe and Levine (1982b) demonstrate that the situation is essentially the same as that in a model with only one good in each period: With nominal initial conditions there is at most a one dimensional indeterminacy and with real initial conditions no indeterminacy is possible. These results are, of course, closely related to those of Balasko and Shell (1981) cited earlier.

IV. An Example with Three Period Lived Consumers

Consider now an economy in which each generation consists of a single consumer who lives three periods and in which there is a single good in each

period. Balasko, Cass, and Shell (1980) present a simple procedure for converting such a model into one in which consumers live two periods. The essential feature of the two period lived model is that each generation overlaps with only one other in each time period. Redefine generations so that generations -1 and 0 become generation 0, generations 1 and 2 become generation 1, and, in general, generations t and t + 1 become generation (t+1)/2 when t is odd. Similarly redefine time periods. The following table is an incidence matrix with a 1 where the generation is alive and a 0 where it is not.

	Time Period 1	2	3	4	5	6	. . .	
Generation -1	1	0	0	0	0	0	. . .	} 0
0	1	1	0	0	0	0	. . .	
1	1	1	1	0	0	0	. . .	} 1
2	0	1	1	1	0	0	. . .	
3	0	0	1	1	1	0	. . .	} 2
4	0	0	0	1	1	1	. . .	
5	0	0	0	0	1	1	. . .	} 3
6	0	0	0	0	0	1	. . .	
.	.	.	.	.	.	.		
.	.	.	.	.	.	.		
.	.	.	.	.	.	.		
	1		2		3			

Notice that now there are two consumers in each redefined generation and two goods in each redefined period, and that each generation overlaps with only one other during any period.

In this section we consider examples of economies in which each consumer lives three periods. Our above discussion indicates that such examples are also examples of an economy in which each consumer lives two periods. Suppose the single consumer in each generation has a utility function of the form

$$u(c_1, c_2, c_3) = (1/b)(a_1 c_1^b + a_2 c_2^b + a_3 c_3^b) \tag{14}$$

where $a_1, a_2, a_3 > 0$ and $b < 1$. This is, of course, the familiar constant elasticity of substitution utility function with elasticity of substitution $\eta = 1/(1 - b)$. The excess demand functions for the consumer born in period t have the form

$$x_j(p_t, p_{t+1}, p_{t+2}) = \frac{a_j^\eta \, \Sigma_{i=1}^3 p_{t+i-1} w_i}{p_{t+j-1}^\eta \, \Sigma_{i=1}^3 a_i^\eta p_{t+i-1}^{1-\eta}} - w_j, \quad j = 1, 2, 3. \tag{15}$$

The equilibrium conditions for such an economy take the form

$$x_3^{-1}(p_1) + x_2^0(p_1, p_2) + x_1(p_1, p_2, p_3) = 0 \tag{16}$$

for $t = 1$,

$$x_3^0(p_1, p_2) + x_2(p_1, p_2, p_3) + x_1(p_2, p_3, p_4) = 0 \tag{17}$$

for t = 2, and

$$x_3(p_{t-2}, p_{t-1}, p_t) + x_2(p_{t-1}, p_t, p_{t+1}) + x_1(p_t, p_{t+1}, p_{t+2}) = 0 \qquad (18)$$

for t > 2. When we redefine generations, time periods, and goods in the manner described above these conditions become the same as (3) and (4). Linearizing (18) around a steady state β produces

$$\beta^2 D_1 x_3 p_{t-2} + (\beta^2 D_2 x_3 + \beta D_1 x_2) p_{t-1} + (\beta^2 D_3 x_3 + \beta D_2 x_2 + D_1 x_1) p_t$$

$$+ (\beta D_3 x_2 + D_2 x_1) p_{t+1} + D_3 x_1 p_{t+2} = 0. \qquad (19)$$

Here all derivatives are evaluated at $(1, \beta, \beta^2)$. We are interested in the roots of the corresponding fourth order polynomial. These correspond, in turn, to eigenvalues of a 4 x 4 matrix like that in (9).

Notice that, in addition to price paths of the form $\{1, \beta, \beta^2, \beta^3, \beta^4, \ldots\}$, the model with two period lived consumers may have steady states of the form $\{1, \gamma, \gamma\delta, \gamma^2\delta, \gamma^2\delta^2, \ldots\}$: When we redefined time periods and goods two period cycles became steady states. In this section, however, we analyze only steady states of the original three period model. Indeed, our numerical examples do not happen to have such two period cycles.

Consider first an example with the following parameter values:

	Period		
	1	2	3
a_i	2	2	1
w_i	3	15	2

$$b = -4$$

This economy has one nominal steady state and three real steady states. The diagram corresponding to that in Figure 1 is given in Figure 2.

Figure 2

The roots of the fourth order difference equation (19) at these steady states are listed below:

	β	Other Roots		
1	0.04239	1.0	0.47907	-0.01380
2	0.93295	1.0	0.17245 ± 0.82735i	
3	1.0	0.93286	0.18594 ± 0.89862i	
4	53.80562	1.0	2.04076	-121.45064

The modulus of the pair of complex conjugates at the steady state where $\beta = 0.93295$ is 0.84513; where $\beta = 1.0$ it is 0.91766. Let us focus our attention on these two steady states. By constructing suitable consumers born in periods -1 and 0, we can find a robust example of an economy with a continuum of equilibria converging to each of these steady states.

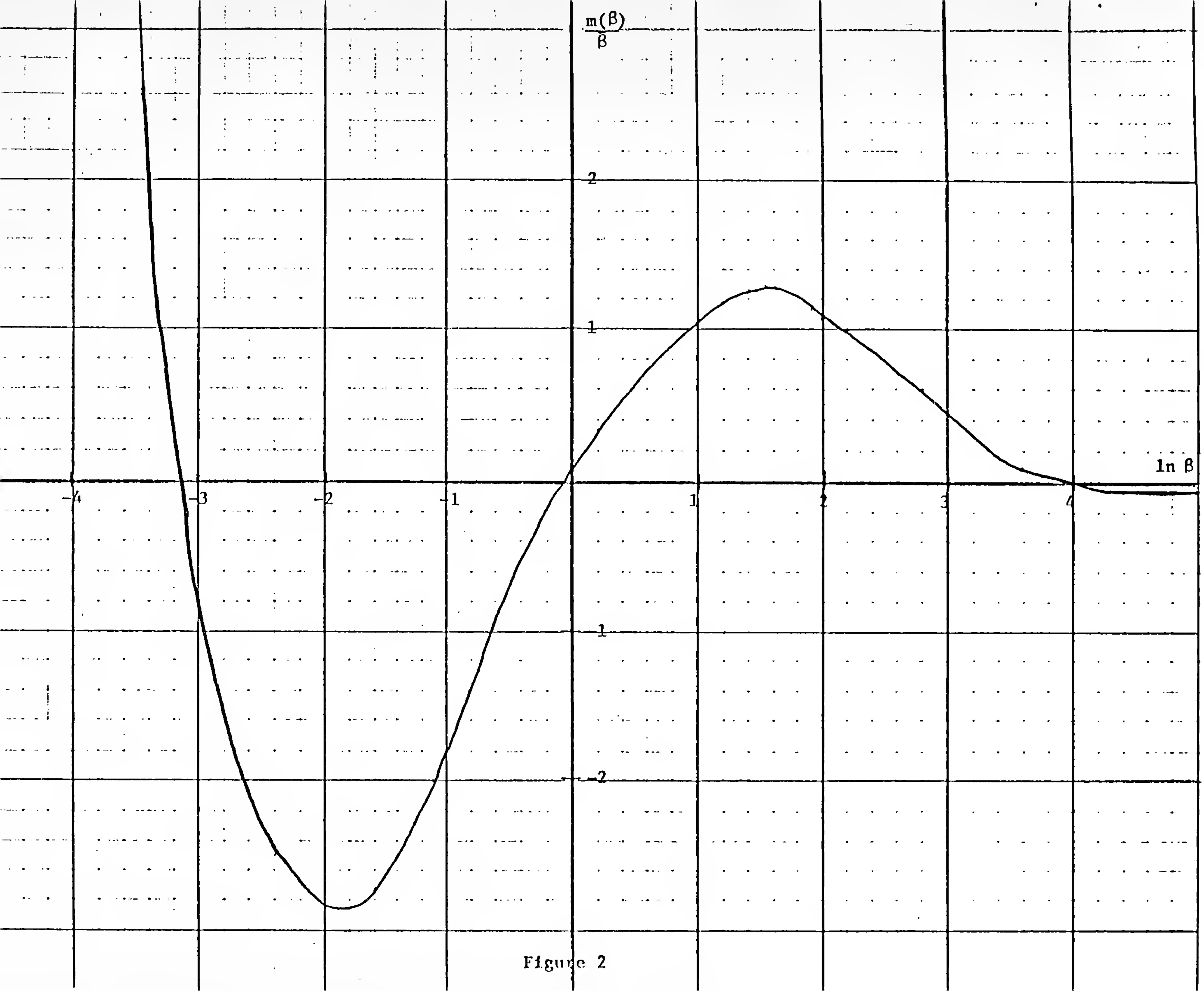

Figure 2

Let the consumer born in period -1 have an endowment of 2 units of the good in his last period of life. He derives utility only from the consumption of this good, of course, so we need not specify any utility function. Let the consumer born in period 0 have an endowment of 10.37894 in the second period of his life and 2 in the third. Let his utility function be $u(c_2, c_3) = -1/4(2c_2^{-4} + c_3^{-4})$. Suppose, in addition, that consumers hold μ^{-1} and μ^{0} in nominal claims respectively at the beginning of the first period. First, consider the case where $\mu^{-1} = \mu^{0} = 0$. It is straightforward, but tedious, to check that $(p_1, p_2, p_3, p_4) = (1, 0.93295, (0.93295)^2, (0.93295)^3)$ satisfies the conditions for equilibrium in the first two periods, (16) and (17). Since $\beta = 0.93295$ is a steady state, this is a legitimate equilibrium price path. Our arguments from the previous section imply that this is only one of a continuum. Since $\mu = 0$, the excess demands of generations -1 and 0 are homogeneous of degree zero and we can normalize prices by setting $p_1 = 1$. We can then choose $p_2 = 0.93295 + \varepsilon$ for any small ε, positive or negative, and use the equilibrium conditions (16) and (17) to solve for p_3 and p_4. Using the equilibrium condition (18), we can solve for an infinite price sequence. This price sequence must converge to one where $p_{t+1}/p_t = 0.93295$ since the modulus of the root governing stability is less than 0.93295. The root equal to unity is, as we have explained, irrelevant since $\mu = 0$ everywhere along this price path.

Now consider the case where $\mu^{-1} = -0.55567$ and $\mu^{0} = 0.65376$. Here it can be checked that $(p_1, p_2, p_3, p_4) = (1, 1, 1, 1)$ satisfies (16) and (17). In this case the excess demands of generations -1 and 0 are not homogeneous and we are not permitted a price normalization: Money is itself the numeraire. We can now choose $p_1 = 1 + \varepsilon_1$ and $p_2 = 1 + \varepsilon_2$ for any $\varepsilon_1, \varepsilon_2$ small enough and use (16) and (17) to solve for p_3 and p_4. Again using (18), we can solve for

an equilibrium price sequence that converges to one where $p_{t+1}/p_t = 1$.

The relative price indeterminacy exhibited in this example does not depend on one of the old consumers coming into the first period with nominal debts, negative fiat money. Suppose, for example, that $\mu^{-1} = 0$, that $\mu^0 = 0.09809$ and that the endowment of the consumer 0 in the second period of his life is 10.93461, but that otherwise consumers -1 and 0 are the same as above. The corresponding demand functions give rise to equilibrium conditions in the first period that are satisfied by $p_t = 1$, $t = 1, 2, 3, 4$. Again there is a two dimensional indeterminacy. Setting $\mu^{-1} = \mu^0 = 0$ does not, however, result in equilibrium conditions that are satisfied by $p_t = (0.93295)^{t-1}$, $t = 1, 2, 3, 4$.

Notice that this example also generates equilibria of the type Gale describes: Any equilibrium price path that converges to the steady state where $\beta = 0.04239$ is determinate and pareto efficient. Any equilibrium price path that converges to the steady state where $\beta = 53.80562$ and has no net nominal debt is also determinate. Any path with non-zero nominal debt that converges to this steady state has a one dimensional indeterminacy, however. Fixing p_1 and μ, that is, fixing the price of fiat money, eliminates this indeterminacy. Any path that converges to this steady state is, of course, pareto inefficient.

Choosing the parameters of this type of model suitably, we can illustrate other possibilities for behavior of equilibrium price paths near steady states. For example, the following parameter values correspond to an economy with four steady states with β's and other roots that are the reciprocals of those given above.

	Period 1	2	3
a_i	1	2	2
w_i	2	15	3

$$b = -4$$

Here the steady state where $\beta = 1$ is unstable: There are no paths that can approach it unless, by pure chance, $p_t = 1$, $t = 1, 2, 3, 4$, satisfies the equilibrium conditions in the first two periods. The steady state where $\beta = 1.07187 = (0.93295)^{-1}$ is also unstable for price paths with no nominal debts. There are, however, determinate price paths with non-zero nominal debt that converge to this steady state.

Suppose that the parameters of the economy are as follows:

	Period 1	2	3
a_i	3	1	1
w_i	4	36	1

$$b = -2$$

This economy has a steady state where $\beta = 2.36512$ and the other roots are 1.0 and $.81499 \pm 1.18782i$. The modulus of this pair of complex conjugates is 1.44053, which is less than β. By choosing the initial old consumers appropriately, we can find a two dimensional continuum of price paths with non-zero nominal debt, and a one dimensional continuum of paths without nominal debts, that converge to this steady state.

An essential feature of all of the above examples is that they are robust: We can perturb slightly the parameters, and even functional forma, of the demand functions of all of the consumers, including the initial old, and still have an economy whose equilibria have the same qualitative features. We choose initial old consumers so that the steady state prices satisfy the equilibrium conditions in the first two periods only to make it easy to verify that there are prices that satisfy these equilibrium conditions and also converge to the steady state.

V. Discussion

We have analyzed the behavior of equilibrium price paths near the steady states of stationary pure exchange overlapping generations models. We have also presented robust examples of such models with three period lived consumers and various forms of relative price indeterminacy. As we have indicated, these examples can be interpreted as models with two period lived consumers and two consumers, rather than one, in each generation and two goods, rather than one, in each period. In this section we discuss some possible extensions of our results and some of their implications for applied work.

Let us first explain how our results can be extended to models with a constant rate of population growth. Suppose the demands of generations t are:

$$\begin{aligned} y_t(p_t,\ p_{t+1}) &= \alpha^{t-1} y_1(p_t,\ p_{t+1}) \\ z_t(p_t,\ p_{t+1}) &= \alpha^{t-1} z_1(p_t,\ p_{t+1}). \end{aligned} \tag{20}$$

Here $\alpha - 1$ is the rate of population growth. With a suitable redefinition of prices and excess demand functions, this model can be transformed into the one we have been working with: Let $\tilde{p}_t = \alpha^{t-1} p_t$, $\tilde{y}(p_t, p_{t+1}) = \alpha y_1(p_t, \alpha p_{t+1})$ and $\tilde{z}(p_t, p_{t+1}) = z(p_t, p_{t+1})$. Notice that $\tilde{y}$ and $\tilde{z}$ are homogeneous of degree zero if y_1 and z_1 are. Notice too that, if y_1 and z_1 satisfy Walras's law,

$$p_t' y_1(p_t, p_{t+1}) + p_{t+1}' z_1(p_t, p_{t+1}) = 0, \tag{21}$$

then so do y and z:

$$\begin{aligned} 0 &= \alpha^{1-t}\tilde{p}_t' y_1(\alpha^{t-1}\tilde{p}_t, \alpha^{-t}\tilde{p}_{t+1}) + \alpha^{-t}\tilde{p}_{t+1}' \tilde{z}(\alpha^{1-t}\tilde{p}_t, \alpha^{-t}\tilde{p}_{t+1}) \\ &= \tilde{p}_t'\tilde{y}(\tilde{p}_t, \tilde{p}_{t+1}) + \tilde{p}_{t+1}'\tilde{z}(\tilde{p}_t, \tilde{p}_{t+1}). \end{aligned} \tag{22}$$

Finally, notice that if p_t, $t = 1, 2, \ldots$, satisfies the equilibrium conditions

$$0 = z_{t-1}(p_{t-1}, p_t) + y_t(p_t, p_{t+1}), \tag{23}$$

then $\tilde{p}_t$, $t = 1, 2, \ldots$, satisfies the corresponding conditions:

$$\begin{aligned} 0 &= \alpha^{t-2} z_1(\alpha^{2-t}\tilde{p}_{t-1}, \alpha^{1-t}\tilde{p}_t) + \alpha^{t-1} y_1(\alpha^{1-t}\tilde{p}_t, \alpha^{-t}\tilde{p}_{t+1}) \\ &= z(\tilde{p}_{t-1}, \tilde{p}_t) + \tilde{y}(\tilde{p}_t, \tilde{p}_{t+1}) \end{aligned} \tag{24}$$

This transformation is obviously invertible: If we know $\tilde{p}_t$, $\tilde{y}$ and $\tilde{z}$, and the growth factor α, we can recover p_t, y_t, and z_t. Nominal steady states are those where $\tilde{p}_t = \tilde{p}_{t+1}$, which is equivalent to $p_t = \alpha p_{t+1}$. This implies Samuelson's result that the rate of interest at such a steady state is, in fact, the growth rate of the population.

Other forms of non-stationarity can be incorporated into our framework as long as the model is stationary for all generations after some generation T. In this case the equilibrium conditions for the first T + 1 periods serve the same role that the equilibrium conditions for the first period do in the stationary model. Generically, they determine all of the price vectors $p_1, p_2, \ldots, p_{T+1}$ but one. The remaining price vector may, or may not, be determined by the condition that P_T and p_{T+1} give rise to a price path that converges to a steady state when viewed as initial values for the difference equation corresponding to the remaining equilibrium conditions. The analysis of relative price indeterminacy remains the same.

A restrictive aspect of our analysis is that we have only analyzed price paths near steady states. In fact, however, our analysis immediately extends to price paths near any cycle of finite length. Recall that when we redefined generations, time periods, and goods to convert the three period lived model into a two period lived model we noted that two cycles become steady states. In general, suppose that a model has a cycle of length k, in other words, that $(p_{t+1}, p_{t+2}, \ldots, p_{t+k}) = \beta(p_{t-k+1}, p_{t-k+2}, \ldots, p_t)$ satisfies the equilibrium conditions. Suppose we redefine generations, time periods and goods so that, for example, generations 1, 2, ..., k are now generation 1 and so on. The cycle now corresponds to a steady state of the redefined model.

Further research is currently being carried out to determine whether our results extend to models with production, durable assets, and some infinitely lived agents. Muller (1983) has extended our analysis to economies with activity analysis production technologies that permit storage of goods from period to period. Woodford (1983) has analyzed models in which there are some finitely lived consumers of the sort we have analyzed here and some infinitely lived consumers. Both find that there remain robust examples of economies with relative price indeterminacy. Interestingly enough, there are also examples where the indeterminacy disappears because of production or infinitely lived agents.

More work obviously remains to be done, but it seems that indeterminacy of perfect foresight equilibria remains a robust characteristic of certain intertemporal equilibrium models in that no small changes in specification can eliminate it. One crucial hypothesis to be examined is that of perfect foresight. With adaptive expectations, for example, equilibrium price paths are generically determinate. Is there some general, and economically meaningful, way to choose a perfect foresight expectations mechanism that gives rise to determinate equilibria? If not, how far do we have to depart from the perfect foresight hypothesis to get determinacy?

Our results should be troubling to researchers interested in applications of the overlapping generations model. A model that does not give determinate results is not very useful for doing policy analysis. One way to handle this type of model is to solve for prices that satisfy the equilibrium conditions for the first T periods by fixing expectations of what prices would be in period T + 1. We could, for example, require that $p_{T+1} = p_T$ or that $p_{T+1} = p$, the steady state price vector. This type of

truncated model generically has determinate equilibria. Furthermore, if the truncation date T is large enough, then an equilibrium of the truncated model would be a reasonable approximation to an equilibrium of the actual model. (How large a T is necessary in practice is itself an interesting question.) Unfortunately, the results may depend crucially on the arbitrarily specified expectations for p_{T+1}: If there is a continuum of equilibrium price paths converging to the steady state, then setting $p_{T+1} = p$ we may get drastically different results, even for p_1, as we vary T.

That the overlapping generations model seems plagued by indeterminacy is not a satisfactory justification for completely abandoning it in favor of the model with a finite number of infinitely lived consumers. As Gale (1973) has pointed out, "the reason for considering a population rather than a fixed set of agents is that the former is what in reality we have, the latter is what we have not." To build a useful intertemporal equilibrium model, however, it would seem necessary to address the issues that we have raised in this paper.

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